SEEMOUS 2016

South Eastern European
Mathematical Olympiad for University Students
Protaras, Cyprus
1-6 March 2016

Mathematical Society of South Eastern Europe
Cyprus Mathematical Society

LANGUAGE: ENGLISH

COMPETITION PROBLEMS

Do all problems 1-4. Each problem is worth 10 points. All answers are written in the booklet provided, following the rules written in the Olympiad programme.

Problem1.

Let f be a continuous and decreasing real valued function, defined on $\left[0,\frac{\pi}{2}\right]$.

Prove the inequalities

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)dx \leq \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx \leq \int_0^1 f(x)dx$$

when do equalities hold?

Problem 2.

a) Prove that for every matrix $X \in M_2(\mathbb{C})$ there exists a matrix $Y \in M_2(\mathbb{C})$ such that $Y^3 = X^2$.
b) Prove that there exists a matrix $A \in M_3(\mathbb{C})$ such that $Z^3 \neq A^2$ for all $Z \in M_3(\mathbb{C})$.

Problem3.

Let $A_1, A_2, \ldots, A_k$ be idempotent matrices ($A_i^2 = A_i$) in $M_n(\mathbb{R})$. Prove that

$$\sum_{i=1}^{k} N(A_i) \geq \operatorname{rank}\left(I - \prod_{i=1}^{k} A_i\right)$$

where $N(A_i) = n - \operatorname{rank}(A_i)$ and $M_n(\mathbb{R})$ is the set of square $n \times n$ matrices with real entries.

Problem4.

Let $n \geq 1$ be an integer and let

$$I_n = \int_0^\infty \frac{\arctan x}{(1+x^2)^n}\,dx$$

Prove that

a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6}$

b) $\displaystyle\int_0^\infty \arctan x \cdot \ln\left(1+\frac{1}{x^2}\right)dx = \frac{\pi^2}{6}$