The 27th Balkan Mathematical Olympiad
Chisinau, Republic of Moldova, May 4 2010

English version

PROBLEMS

Each problem is worth 10 points.

Time allowed is 4 hours 30 min.

Problem 1. Let \(a, b\) and \(c\) be positive real numbers. Prove that
\[
\frac{a^2 b(b - c)}{a + b} + \frac{b^2 c(c - a)}{b + c} + \frac{c^2 a(a - b)}{c + a} \geq 0
\]

Problem 2. Let \(ABC\) be an acute triangle with orthocenter \(H\), and let \(M\) be the midpoint of \(AC\). The point \(C_1\) on \(AB\) is such that \(CC_1\) is an altitude of the triangle \(ABC\). Let \(H_1\) be the reflection of \(H\) in \(AB\). The orthogonal projections of \(C_1\) onto the lines \(AH_1\), \(AC\) and \(BC\) are \(P\), \(Q\) and \(R\), respectively. Let \(M_1\) be the point such that the circumcentre of triangle \(PQR\) is the midpoint of the segment \(MM_1\).
Prove that \(M_1\) lies on the segment \(BH_1\).

Problem 3. A strip of width \(w\) is the set of all points which lie on, or between, two parallel lines distance \(w\) apart. Let \(S\) be a set of \(n\) \((n \geq 3)\) points on the plane such that any three different points of \(S\) can be covered by a strip of width \(1\).
Prove that \(S\) can be covered by a strip of width \(2\).

Problem 4. For each integer \(n\) \((n \geq 2)\), let \(f(n)\) denote the sum of all positive integers that are at most \(n\) and not relatively prime to \(n\).
Prove that \(f(n + p) \neq f(n)\) for each such \(n\) and every prime \(p\).