05-10 May 2016 Tirana, Albania

Language: **English**

The 33^{rd} Balkan Mathematical Olympiad
Tirana, May 7, 2016

Problem 1.

Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i\Big(f(x+i+1) - f(f(x+i))\Big)\right| < 2016.$$

Solution. From the condition of the problem we get

$$\left|\sum_{i=1}^{n-1} i\Big(f(x+i+1) - f(f(x+i))\Big)\right| < 2016 \tag{1}$$

Then

$$\begin{aligned} &\left|n\Big(f(x+n+1) - f(f(x+n))\Big)\right| \\ =&\left|\sum_{i=1}^{n} i\Big(f(x+i+1) - f(f(x+i))\Big) - \sum_{i=1}^{n-1} i\Big(f(x+i+1) - f(f(x+i))\Big)\right| \\ <&2 \cdot 2016 = 4032 \end{aligned} \tag{2}$$

implying

$$|f(x+n+1) - f(f(x+n))| < \frac{4032}{n} \tag{3}$$

for every real number x and every positive integer n.

Let $y \in \mathbb{R}$ be arbitrary. Then there exists x such that $y = x + n$. We obtain

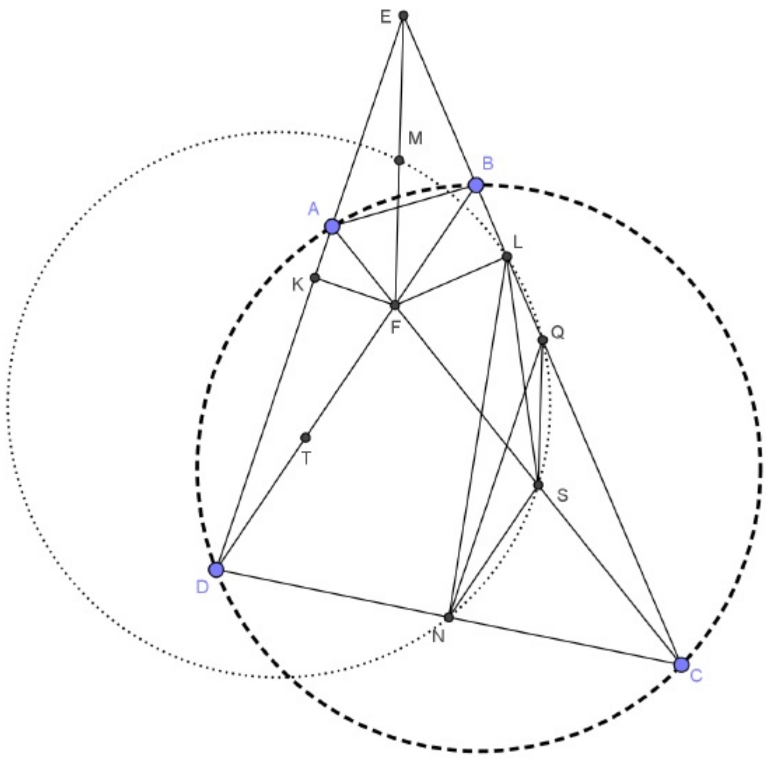
$$|f(y+1) - f(f(y))| < \frac{4032}{n} \tag{4}$$

for every real number y and every positive integer n. The last inequality holds for every positive integer n from where $f(y+1) = f(f(y))$ for every $y \in \mathbb{R}$ and since the function f is an injection, then $f(y) = y + 1$. The function $f(y) = y + 1$ satisfies the required condition. $\square$

Problem 2.

Let $ABCD$ be a cyclic quadrilateral with $AB < CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.

Solution. Let N be the midpoint of CD. We will prove that the circumcircles of the triangles MKT and MLS pass through N. (1)
First will prove that the circumcircle of MLS passes through N.
Let Q be the midpoint of EC. Note that the circumcircle of MLS is the **Euler circle** (2) of the triangle EFC, so it passes also through Q. (*)(3)



We will prove that

$$\angle SLQ = \angle QNS \qquad \text{or} \qquad \angle SLQ + \angle QNS = 180^\circ \tag{4}$$

Indeed, since FLC is right-angled and LS is its median, we have that $SL = SC$ and

$$\angle SLC = \angle SCL = \angle ACB \tag{5}$$

In addition, since N and S are the midpoints of DC and FC we have that $SN \parallel FD$ and similarly, since Q and N are the midpoints of EC and CD , so $QN \parallel ED$.
It follows that the angles $\angle EDB$ and $\angle QNS$ have parallel sides, and since $AB < CD$, they are acute, and as a result we have that

$$\angle EDB = \angle QNS \qquad \text{or} \qquad \angle EDB + \angle QNS = 180^\circ \tag{6}$$

But, from the cyclic quadrilateral $ABCD$, we get that

$$\angle EDB = \angle ACB \tag{7}$$

Now, from (2),(3) and (4) we obtain immediately (1), so the quadrilateral $LNSQ$ is cyclic. Since from (*), its circumcircle passes also through M, we get that the points M, L, Q, S, N are cocylic and this means that the circumcircle of MLS passes through N.
Similarly, the circumcircle of MKT passes also through N and we have the desired. $\square$

Problem 3.

Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2\left(f(p)!\right)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

Note: A monic polynomial has leading coefficient equal to 1.

Solution. If f is a constant polynomial then it's obvious that the condition cannot hold for

$$p \geq 5 \quad \text{since} \quad f(p)=1 \tag{1}$$

From the divisibility relation $p|2\left(f(p)\right)!+1$ we conclude that:

$$f(p)<p, \quad \text{for all primes} \quad p>N \qquad (*) \tag{2}$$

In fact, if for some prime number p we have $f(p) \geq p$, then $p|\left(f(p)\right)!$ and then $p|1$, which is absurd.

Now suppose that $\deg f=m>1$. Then $f(x)=x^m+Q(x), \deg Q(x) \leq m-1$ and so $f(p)=p^m+Q(p)$. Hence for some large enough prime number p holds that $f(p)>p$, which contradicts (*). Therefore we must have $\deg f(x)=1$ and $f(x)=x-a$, for some positive integer a. (3)

Thus the given condition becomes:

$$p|2(p-a)!+1 \tag{4}$$

But we have (using Wilsons theorem)

$$\begin{aligned} 2(p-3)! \equiv -(p-3)!(p-2) \equiv -(p-2)! \equiv -1 \pmod{p} \\ \Rightarrow p|2(p-3)!+1 \end{aligned} \tag{5}$$

From (1) and (2) we get

$$\begin{aligned} &(p-3)! \equiv (p-a)! \pmod{p} \\ &(p-3)!(-1)^a(a-1)! \equiv (p-a)!(-1)^a(a-1)! \pmod{p} \\ &(p-3)!(-1)^a(a-1)! \equiv 1 \pmod{p} \end{aligned}$$

Since $-2(p-3)! \equiv 1 \pmod{p}$, it follows that

$$(-1)^a(a-1)! \equiv -2 \pmod{p} \tag{6}$$

Taking $p>(a-1)!$, we conclude that $a=3$ and so $f(x)=x-3$, for all x.

The function $f(x)=x-3$ satisfies the required condition. $\square$

Problem 4.

The plane is divided into unit squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note: Any rectangle is assumed here to have sides contained in the lines of the grid.

Solution. Let the centers of the unit squares be the integer points in the plane, and denote each unit square by the coordinates of its center.

Consider the set D of all unit squares (x, y) such that $|x| + |y| \leq 24$. Any integer translate of D is called a *diamond*.

Since any two unit squares that belong to the same diamond also belong to some rectangle of perimeter 100, a diamond cannot contain two unit squares of the same colour. Since a diamond contains exactly $24^2 + 25^2 = 1201$ unit squares, a diamond must contain every colour exactly once.

Choose one colour, say, green, and let a_1, a_2, $\ldots$ be all green unit squares. Let P_i be the diamond of center a_i. We will show that no unit square is covered by two P's and that every unit square is covered by some P_i.

Indeed, suppose first that P_i and P_j contain the same unit square b. Then their centers lie within the same rectangle of perimeter 100, a contradiction.

Let, on the other hand, b be an arbitrary unit square. The diamond of center b must contain some green unit square a_i. The diamond P_i of center a_i will then contain b.

Therefore, P_1, P_2, $\ldots$ form a covering of the plane in exactly one layer. It is easy to see, though, that, up to translation and reflection, there exists a unique such covering. (Indeed, consider two neighbouring diamonds. Unless they fit neatly, uncoverable spaces of two unit squares are created near the corners: see Fig. 1.)

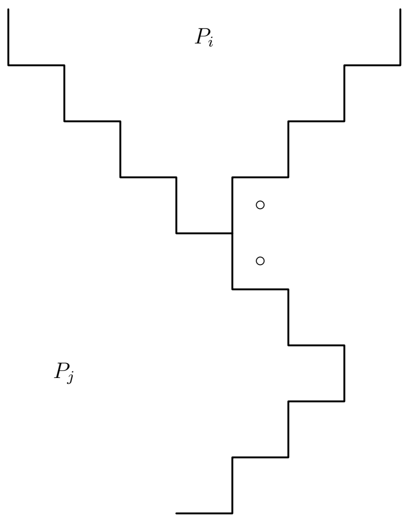


Figure 1:

Without loss of generality, then, this covering is given by the diamonds of centers (x, y) such that $24x+25y$ is divisible by 1201. (See Fig. 2 for an analogous covering with smaller diamonds.) It follows from this that no rectangle of size 1×1201 can contain two green unit squares, and analogous reasoning works for the remaining colours.

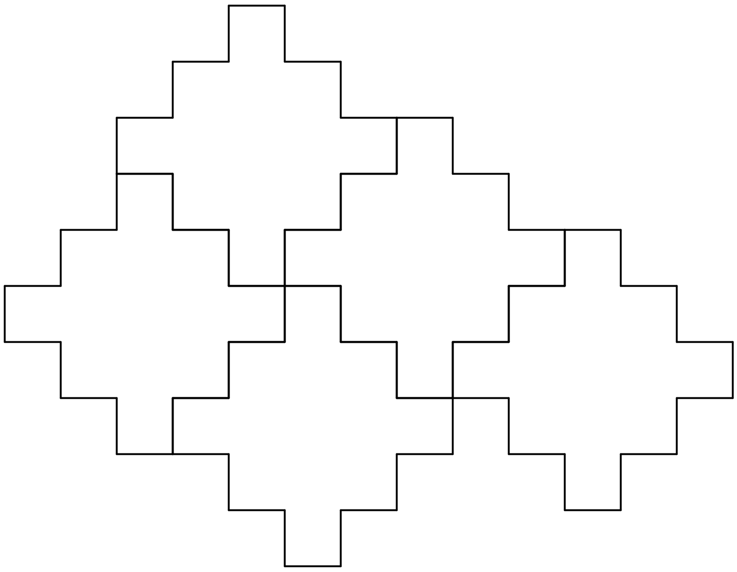


Figure 2:

$\square$