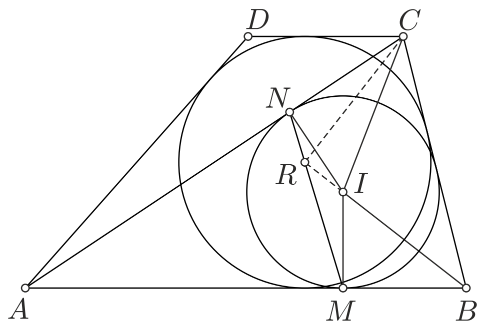
JBMO 2016
Problems and solutions

Problem 1. A trapezoid $ABCD$ ($AB \parallel CD$, $AB > CD$) is circumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.



Solution.
Version 1. Let I be the incenter of triangle ABC and R be the common point of the lines BI and MN. Since

$$m(\widehat{ANM}) = 90^\circ - \frac{1}{2}m(\widehat{MAN}) \quad \text{and} \quad m(\widehat{BIC}) = 90^\circ + \frac{1}{2}m(\widehat{MAN})$$

the quadrilateral $IRNC$ is cyclic. (1)

It follows that $m(\widehat{BRC}) = 90^\circ$ and therefore

$$m(\widehat{BCR}) = 90^\circ - m(\widehat{CBR}) = 90^\circ - \frac{1}{2}\big(180^\circ - m(\widehat{BCD})\big) = \frac{1}{2}m(\widehat{BCD}). \qquad (2)$$

So, $(CR$ is the angle bisector of $\widehat{DCB}$ and R is the incenter of the trapezoid. (3)

Version 2. If R is the incentre of the trapezoid $ABCD$, then B, I and R are collinear, (1')
and $m(\widehat{BRC}) = 90^\circ$. (2')
The quadrilateral $IRNC$ is cyclic. (3')
Then $m(\widehat{MNC}) = 90^\circ + \frac{1}{2} \cdot m(\widehat{BAC})$ (4')
and $m(\widehat{RNC}) = m(\widehat{BIC}) = 90^\circ + \frac{1}{2} \cdot m(\widehat{BAC})$, (5')

so that $m(\widehat{MNC}) = m(\widehat{RNC})$ and the points M, R and N are collinear. (6')

Version 3. If R is the incentre of the trapezoid $ABCD$, let $M' \in (AB)$ and $N' \in (AC)$ be the unique points, such that $R \in M'N'$ and $(AM') \equiv (AN')$. (1")
Let S be the intersection point of CR and AB. Then $CR = RS$. (2")
Consider $K \in AC$ such that $SK \parallel M'N'$. Then N' is the midpoint of (CK). (3")
We deduce

$$AN' = \frac{AK + AC}{2} = \frac{AS + AC}{2} = \frac{AB - BS + AC}{2} = \frac{AB + AC - BC}{2} = AN. \qquad (4'')$$

We conclude that $N = N'$, hence $M = M'$, and R, M, N are collinear. (5")

□

Problem 2. Let a, b and c be positive real numbers. Prove that

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(b+c)^2+4abc}+\frac{8}{(c+a)^2+4abc}+a^2+b^2+c^2 \geq \frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}\ .$$

Solution. Since $2ab \leq a^2+b^2$, it follows that $(a+b)^2 \leq 2(a^2+b^2)$ (1)
and $4abc \leq 2c(a^2+b^2)$, for any positive reals a, b, c. (2)
Adding these inequalities, we find

$$(a+b)^2+4abc \leq 2(a^2+b^2)(c+1), \tag{3}$$

so that

$$\frac{8}{(a+b)^2+4abc} \geq \frac{4}{(a^2+b^2)(c+1)}. \tag{4}$$

Using the AM-GM inequality, we have

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2} \geq 2\sqrt{\frac{2}{c+1}}=\frac{4}{\sqrt{2(c+1)}}\ , \tag{5}$$

respectively

$$\frac{c+3}{8}=\frac{(c+1)+2}{8} \geq \frac{\sqrt{2(c+1)}}{4}\ . \tag{6}$$

We conclude that

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2} \geq \frac{8}{c+3}\ , \tag{7}$$

and finally

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(a+c)^2+4abc}+\frac{8}{(b+c)^2+4abc}+a^2+b^2+c^2 \geq \frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}\ . \tag{8}$$

□

Problem 3. Find all the triples of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016.
(*A power of* 2016 *is an integer of the form* 2016^n, *where* n *is a non-negative integer.*)

Solution. Let a, b, c be integers and n be a positive integer such that

$$(a-b)(b-c)(c-a) + 4 = 2 \cdot 2016^n.$$

We set $a - b = -x$, $b - c = -y$ and we rewrite the equation as

$$xy(x+y) + 4 = 2 \cdot 2016^n. \qquad (1)$$

If $n > 0$, then the right hand side is divisible by 7, so we have that

$$xy(x+y) + 4 \equiv 0 \pmod 7 \qquad (2)$$

or

$$3xy(x+y) \equiv 2 \pmod 7 \qquad (3)$$

or

$$(x+y)^3 - x^3 - y^3 \equiv 2 \pmod 7. \qquad (4) \qquad (1)$$

Note that, by Fermat's Little Theorem, for any integer k the cubic residues are $k^3 \equiv -1, 0, 1 \pmod 7$. (5)
It follows that in (1) some of $(x+y)^3$, x^3 and y^3 should be divisible by 7.
But in this case, $xy(x+y)$ is divisible by 7 and this is a contradiction. (6)

So, the only possibility is to have $n = 0$ and consequently, $xy(x+y) + 4 = 2$, or, equivalently, $xy(x+y) = -2$. (7)
The solutions for this are $(x, y) \in \{(-1,-1), (2,-1), (-1,2)\}$, (8)
so the required triples are $(a, b, c) = (k+2, k+1, k)$, $k \in \mathbb{Z}$, and all their cyclic permutations. (9)

Alternative version: If $n > 0$ then 9 divides $(a-b)(b-c)(c-a) + 4$, that is, the equation $xy(x+y) + 4 \equiv 0 \pmod 9$) has the solution $x = b - a, y = c - b$. (1')
But then x and y have to be 1 modulo 3, implying $xy(x+y) \equiv 2 \pmod 9$, which is a contradiction. (2')
We can continue now as in the first version.

$\square$

Problem 4. A 5×5 table is called *regular* if each of its cells contains one of four pairwise distinct real numbers, such that each of them occurs exactly once in every 2×2 subtable. The sum of all numbers of a *regular table* is called the *total sum* of the table. With any four numbers, one constructs all possible regular tables, computes their total sums and counts the distinct outcomes. Determine the maximum possible count.

Solution. We will prove that the maximum number of total sums is 60.
The proof is based on the following claim.
Claim. In a regular table either each row contains exactly two of the numbers, or each column contains exactly two of the numbers.
Proof of the Claim. Indeed, let R be a row containing at least three of the numbers. Then, in row R we can find three of the numbers in consecutive positions, let x, y, z be the numbers in consecutive positions(where $\{x, y, s, z\} = \{a, b, c, d\}$). Due to our hypothesis that in every 2×2 subarray each number is used exactly once, in the row above R(if there is such a row), precisely above the numbers x, y, z will be the numbers z, t, x in this order. And above them will be the numbers x, y, z in this order. The same happens in the rows below R (see at the following figure).

$$\begin{pmatrix} \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \end{pmatrix}$$

Completing all the array, it easily follows that each column contains exactly two of the numbers and our claim is proven. (1)
Rotating the matrix (if it is necessary), we may assume that each row contains exactly two of the numbers. If we forget the first row and column from the array, we obtain a 4×4 array, that can be divided into four 2×2 subarrays, containing thus each number exactly four times, with a total sum of $4(a + b + c + d)$.
It suffices to find how many different ways are there to put the numbers in the first row R_1 and the first column C_1. (2)
Denoting by a_1, b_1, c_1, d_1 the number of appearances of a, b, c, and respectively d in R_1 and C_1, the total sum of the numbers in the entire 5×5 array will be

$$S = 4(a + b + c + d) + a_1 \cdot a + b_1 \cdot b + c_1 \cdot c + d_1 \cdot d. \qquad (3)$$

If the first, the third and the fifth row contain the numbers x, y, with x denoting the number at the entry $(1, 1)$, then the second and the fourth row will contain only the numbers z, t, with z denoting the number at the entry $(2, 1)$. Then $x_1 + y_1 = 7$ and $x_1 \geqslant 3$, $y_1 \geqslant 2$, $z_1 + t_1 = 2$, and $z_1 \geqslant t_1$. Then $\{x_1, y_1\} = \{5, 2\}$ or $\{x_1, y_1\} = \{4, 3\}$, respectively $\{z_1, t_1\} = \{2, 0\}$ or $\{z_1, t_1\} = \{1, 1\}$. (4)

Then (a_1, b_1, c_1, d_1) is obtained by permuting one of the following quadruples:

$$(5, 2, 2, 0), (5, 2, 1, 1), (4, 3, 2, 0), (4, 3, 1, 1). \qquad (5)$$

There are a total of $\frac{4!}{2!} = 12$ permutations of $(5, 2, 2, 0)$, also 12 permutations of $(5, 2, 1, 1)$, 24 permutations of $(4, 3, 2, 0)$ and finally, there are 12 permutations of $(4, 3, 1, 1)$. Hence, there are at most 60 different possible total sums. (6)

We can obtain indeed each of these 60 combinations: take three rows *ababa* alternating with two rows *cdcdc* to get $(5, 2, 2, 0)$; take three rows *ababa* alternating with one row *cdcdc* and a row (*dcdcd*) to get $(5, 2, 1, 1)$; take three rows *ababc* alternating with two rows *cdcda* to get $(4, 3, 2, 0)$; take three rows *abcda* alternating with two rows *cdabc* to get $(4, 3, 1, 1)$. (7)

By choosing for example $a = 10^3$, $b = 10^2$, $c = 10$, $d = 1$, we can make all these sums different. (8)

Hence, 60 is indeed the maximum possible number of different sums. (9)

Alternative Version: Consider a regular table containing the four distinct numbers a, b, c, d. The four 2×2 corners contain each all the four numbers, so that, if a_1, b_1, c_1, d_1 are the numbers of appearances of a, b, c, and respectively d in the middle row and column, then

$$S = 4(a + b + c + d) + a_1 \cdot a + b_1 \cdot b + c_1 \cdot c + d_1 \cdot d. \qquad (1')$$

Consider the numbers x in position $(3, 3)$, y in position $(3, 2)$, y' in position $(3, 4)$, z in position $(2, 3)$ and z' in position $(4, 3)$.

If $z \neq z' = t$, then $y = y'$, and in positions $(3, 1)$ and $(3, 5)$ there will be the number x. (2')

The second and fourth row can only contain now the numbers z and t, respectively the first and fifth row only x and y. (3').

Then $x_1 + y_1 = 7$ and $x_1 \geqslant 3$, $y_1 \geqslant 2$, $z_1 + t_1 = 2$, and $z_1 \geqslant t_1$. Then $\{x_1, y_1\} = \{5, 2\}$ or $\{x_1, y_1\} = \{4, 3\}$, respectively $\{z_1, t_1\} = \{2, 0\}$ or $\{z_1, t_1\} = \{1, 1\}$. (4')

One can continue now as in the first version.

$\square$