SEEMOUS 2007
South Eastern European Mathematical Olympiad for University Students
Agros, Cyprus
7-12 March 2007

Mathematical Society of South Eastern Europe
Cyprus Mathematical Society

COMPETITION PROBLEMS

9 March 2007

Do all problems 1-4. Each problem is worth 10 points. All answers should be answered in the booklet provided, based on the rules written in the Olympiad programme. Time duration: 9.00 – 14.00

PROBLEM 1

Given $a \in (0,1) \cap \mathbb{Q}$ let $a = 0,a_1 a_2 a_3 \ldots$ be its decimal representation. Define

$$f_a(x) = \sum_{n=1}^{\infty} a_n x^n, \; x \in (0, 1).$$

Prove that f_a is a rational function of the form $f_a(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients.

Conversely, if $a_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \in \mathbb{N}$, and $f_a(x) = \sum_{n=1}^{\infty} a_n x^n$ for $x \in (0, 1)$ is a rational function of the form $f_a(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients, prove that the number $a = 0,a_1 a_2 a_3 \ldots$ is rational.

PROBLEM 2

Let $f(x) = \underbrace{\max}_{i} |x_i|$ for $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and let A be an nxn matrix such that $f(Ax) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that there exists a positive integer m such that A^m is the identity matrix I_n.

PROBLEM 3

Let F be a field and let P: F x F $\to$ F be a function such that for every $x_0 \in F$ the function $P(x_0 , y)$ is a polynomial in y and for every $y_0 \in F$ the function $P(x , y_0)$ is a polynomial in x.
Is it true that P is necessarily a polynomial in x and y, when

a) $F = \mathbb{Q}$, the field of rational numbers?

b) F is a finite field?

Prove your claims.

PROBLEM 4

For $x \in \mathbb{R}$, $y \geq 0$ and $n \in \mathbb{Z}$ denote by $w_n(x , y) \in [0 , \pi)$ the angle in radians with which the segment joining the point (n , 0) to the point (n + y , 0) is seen from the point $(x , 1) \in \mathbb{R}^2$.

a) Show that for every $x \in \mathbb{R}$ and $y \geq 0$, the series $\sum_{n=-\infty}^{\infty} w_n(x,y)$ converges.

If we now set $w(x,y) = \sum_{n=-\infty}^{\infty} w_n(x,y)$, show that $w(x,y) \leq ([y] + 1)\pi$.

([y] is the integer part of y)

b) Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every y with $0 < y < \delta$ and every $x \in \mathbb{R}$ we have $w(x,y) < \varepsilon$.

c) Prove that the function $w : \mathbb{R} \times [0 , +\infty) \to [0 , +\infty)$ defined in (a) is continuous.