PROBLEM 1

Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

Solution. The points E, K, H, G are on the circle of diameter GE, so

$$\angle EHK = \angle EGK. \qquad (\dagger)$$

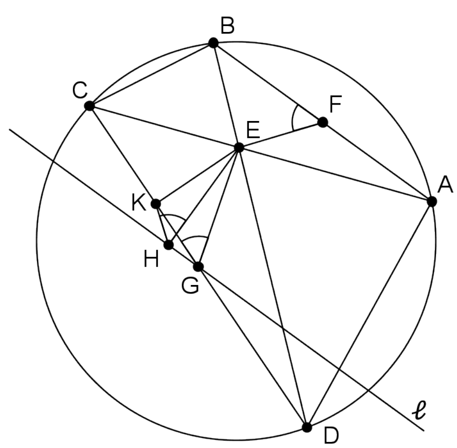
Also, from $\angle DCA = \angle DBA$ and $\frac{CE}{CD} = \frac{BE}{BA}$ it follows

$$\frac{CE}{CG} = \frac{2CE}{CD} = \frac{2BE}{BA} = \frac{BE}{BF},$$

therefore $\Delta CGE \sim \Delta BFE$. In particular, $\angle EGC = \angle BFE$, so by ($\dagger$)

$$\angle EHK = \angle BFE.$$

But $HE \perp FB$ and so, since FE and HK are obtained by rotations of these lines by the same (directed) angle, $FE \perp HK$.



Marking Scheme. $\angle EHK = \angle EGK$ **2p**
Similarity of $\Delta CGE \sim \Delta BFE$ **3p**
$\angle EGC = \angle BFE$ **1p**
$\angle EHK = \angle BFE$ **1p**
Concluding the proof **3p**
Remark. Any partial or equivalent approach should be marked accordingly.

PROBLEM 2

Given real numbers x, y, z such that $x+y+z=0$, show that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1}\geq 0.$$

When does equality hold?

Solution. The inequality is clear if $xyz=0$, in which case equality holds if and only if $x=y=z=0$.

Henceforth assume $xyz\neq 0$ and rewrite the inequality as

$$\frac{(2x+1)^2}{2x^2+1}+\frac{(2y+1)^2}{2y^2+1}+\frac{(2z+1)^2}{2z^2+1}\geq 3.$$

Notice that (exactly) one of the products xy, yz, zx is positive, say $yz>0$, to get

$$\begin{aligned}
\frac{(2y+1)^2}{2y^2+1}+\frac{(2z+1)^2}{2z^2+1} &\geq \frac{2(y+z+1)^2}{y^2+z^2+1} && \text{(by Jensen)}\\
&= \frac{2(x-1)^2}{x^2-2yz+1} && \text{(for } x+y+z=0\text{)}\\
&\geq \frac{2(x-1)^2}{x^2+1}. && \text{(for } yz>0\text{)}
\end{aligned}$$

Here equality holds if and only if $x=1$ and $y=z=-1/2$. Finally, since

$$\frac{(2x+1)^2}{2x^2+1}+\frac{2(x-1)^2}{x^2+1}-3=\frac{2x^2(x-1)^2}{(2x^2+1)(x^2+1)}\geq 0,\quad x\in\mathbb{R},$$

the conclusion follows. Clearly, equality holds if and only if $x=1$, so $y=z=-1/2$. Therefore, if $xyz\neq 0$, equality holds if and only if one of the numbers is 1, and the other two are $-1/2$.

Marking Scheme. Proving the inequality and identifying the equality case when one of the variables vanishes **1p**

Applying Jensen or Cauchy–Schwarz inequality to the fractions involving the pair of numbers of the same sign **3p**

Producing the corresponding lower bound in the third variable **3p**

Proving the resulting one–variable inequality **2p**

Deriving the equality case **1p**

Remark. Any partial or equivalent approach should be marked accordingly.

PROBLEM 3

Let S be a finite set of positive integers which has the following property: if x is a member of S, then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. We agree that a singleton subset of S is both good and bad. Let k be the largest possible size of a good subset of S. Prove that k is also the smallest number of pairwise-disjoint bad subsets whose union is S.

Solution. Notice first that a bad subset of S contains at most one element from a good one, to deduce that a partition of S into bad subsets has at least as many members as a maximal good subset.

Notice further that the elements of a good subset of S must be among the terms of a geometric sequence whose ratio is a prime: if $x < y < z$ are elements of a good subset of S, then $y = xp^\alpha$ and $z = yq^\beta = xp^\alpha q^\beta$ for some primes p and q and some positive integers α and β, so $p = q$ for z/x to be a power of a prime.

Next, let $P = \{2, 3, 5, 7, 11, \cdots\}$ denote the set of all primes, let

$$m = \max\{\exp_p x : x \in S \text{ and } p \in P\},$$

where $\exp_p x$ is the exponent of the prime p in the canonical decomposition of x, and notice that a maximal good subset of S must be of the form $\{a, ap, \cdots, ap^m\}$ for some prime p and some positive integer a which is not divisible by p. Consequently, a maximal good subset of S has $m + 1$ elements, so a partition of S into bad subsets has at least $m + 1$ members.

Finally, notice by maximality of m that the sets

$$S_k = \{x : x \in S \text{ and } \sum_{p \in P} \exp_p x \equiv k \pmod{m+1}\}, \quad k = 0, 1, \cdots, m,$$

form a partition of S into $m + 1$ bad subsets. The conclusion follows.

Marking Scheme. Identification of the structure of a good set **1p**
Considering the maximal exponent m of a prime and deriving $k = m + 1$ **1p**
Noticing that the intersection of a bad set and a good set contains at most one element and infering that a partition of S into bad sets has at least k members..... **2p**
Producing a partition of S into k bad subsets **6p**
Remark. Any partial or equivalent approach should be marked accordingly.

PROBLEM 4

Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$.

Solution. Unless otherwise stated, throughout the proof indices take on values from 0 to 5 and are reduced modulo 6. Label the vertices of the hexagon in circular order, A_0, A_1, $\cdots$, A_5, and let the lines of support of the alternate sides A_iA_{i+1} and $A_{i+2}A_{i+3}$ meet at B_i. To show that the area of at least one of the triangles $B_0B_2B_4$, $B_1B_3B_5$ is greater than or equal to $3/2$, it is sufficient to prove that the total area of the six triangles $A_{i+1}B_iA_{i+2}$ is at least 1:

$$\sum_{i=0}^{5} \text{area } A_{i+1}B_iA_{i+2} \geq 1.$$

To begin with, reflect each B_i through the midpoint of the segment $A_{i+1}A_{i+2}$ to get the points B'_i. We shall prove that the six triangles $A_{i+1}B'_iA_{i+2}$ cover the hexagon. To this end, reflect A_{2i+1} through the midpoint of the segment $A_{2i}A_{2i+2}$ to get the points A'_{2i+1}, $i = 0, 1, 2$. The hexagon splits into three parallelograms, $A_{2i}A_{2i+1}A_{2i+2}A'_{2i+1}$, $i = 0, 1, 2$, and a (possibly degenerate) triangle, $A'_1A'_3A'_5$. Notice first that each parallelogram $A_{2i}A_{2i+1}A_{2i+2}A'_{2i+1}$ is covered by the pair of triangles $(A_{2i}B'_{2i+5}A_{2i+1}, A_{2i+1}B'_{2i}A_{2i+2})$, $i = 0, 1, 2$. The proof is completed by showing that at least one of these pairs contains a triangle that covers the triangle $A'_1A'_3A'_5$. To this end, it is sufficient to prove that $A_{2i}B'_{2i+5} \geq A_{2i}A'_{2i+5}$ and $A_{2j+2}B'_{2j} \geq A_{2j+2}A'_{2j+3}$ for some indices $i, j \in \{0, 1, 2\}$. To establish the first inequality, notice that

$$A_{2i}B'_{2i+5} = A_{2i+1}B_{2i+5}, \quad A_{2i}A'_{2i+5} = A_{2i+4}A_{2i+5}, \quad i = 0, 1, 2,$$

$$\frac{A_1B_5}{A_4A_5} = \frac{A_0B_5}{A_5B_3} \quad \text{and} \quad \frac{A_3B_1}{A_0A_1} = \frac{A_2A_3}{A_0B_5},$$

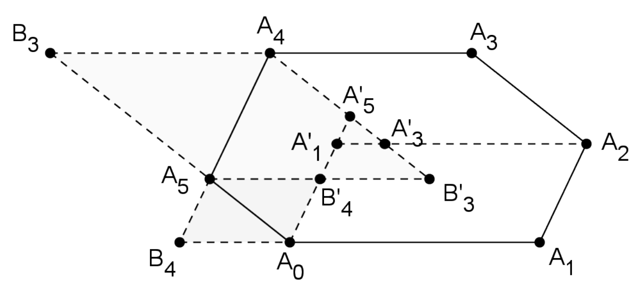
to get

$$\prod_{i=0}^{2} \frac{A_{2i}B'_{2i+5}}{A_{2i}A'_{2i+5}} = 1.$$

Similarly,

$$\prod_{j=0}^{2} \frac{A_{2j+2}B'_{2j}}{A_{2j+2}A'_{2j+3}} = 1,$$

whence the conclusion.



Marking Scheme. Stating that the total area of the small triangles ≥ 1 .... **1p**

Idea of covering the hexagon by flipping the small triangles.................... **2p**

Decomposition of the hexagon into three adequate parallelograms and a triangle **1p**

Proving that each pair of triangles adjacent to a parallelogram covers that parallelogram **1p**

Proving the central triangle also covered **5p**

Remark. Any partial or equivalent approach should be marked accordingly.