# SEEMOUS 2009

**South Eastern European Mathematical Olympiad for University Students**

AGROS, March 6, 2009

COMPETITION PROBLEMS

**Problem 1**

a) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n x^k dx,$$

where $k \in \mathbb{N}$.

b) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x) dx,$$

where $f : [0,1] \to \mathbb{R}$ is a continuous function.

**Solution** Answer: $f\left(\frac{1}{2}\right)$. Proof: Set

$$L_n(f) = \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x)\, dx .$$

A straightforward calculation (integrating by parts) shows that

$$\int_0^1 (x(1-x))^n x^k dx = \frac{(n+k)!n!}{(2n+k+1)!} .$$

Thus, $\displaystyle\int_0^1 (x(1-x))^n dx = \frac{(n!)^2}{(2n+1)!}$ and desired limit is equal to $\lim_{n\to\infty} L_n(f)$. Next,

$$\lim_{n\to\infty} L_n(x^k) = \lim_{n\to\infty} \frac{(n+1)(n+2)\dots(n+k)}{(2n+2)(2n+3)\dots(2n+k+1)} = \frac{1}{2^k} .$$

According to linearity of the integral and of the limit, $\lim_{n\to\infty} L_n(P) = P\left(\frac{1}{2}\right)$ for every polynomial $P(x)$.

Finally, fix an arbitrary $\varepsilon > 0$. A polynomial $P$ can be chosen such that $|f(x) - P(x)| < \varepsilon$ for every $x \in [0,1]$. Then

$$|L_n(f) - L_n(P)| \le L_n(|f-P|) < L_n(\varepsilon \cdot \mathbb{I}) = \varepsilon\, , \text{ where } \mathbb{I}(x) = 1, \text{ for every } x \in [0,1] .$$

There exists $n_0$ such that $\left|L_n(P) - P\left(\frac{1}{2}\right)\right| < \varepsilon$ for $n \geq n_0$. For these integers

$$\left|L_n(f) - f\left(\frac{1}{2}\right)\right| \leq |L_n(f) - L_n(P)| + \left|L_n(P) - P\left(\frac{1}{2}\right)\right| + \left|f\left(\frac{1}{2}\right) - P\left(\frac{1}{2}\right)\right| < 3\varepsilon\,,$$

which concludes the proof.

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**Problem 2**

Let $P$ be a real polynomial of degree five. Assume that the graph of $P$ has three inflection points lying on a straight line. Calculate the ratios of the areas of the bounded regions between this line and the graph of the polynomial $P$.

**Solution** Denote the inflection points by $A$, $B$, and $C$. Let $l : y = kx + n$ be the equation of the line that passes through them. If $B$ has coordinates $(x_0, y_0)$, the affine change

$$x' = x - x_0, \quad y' = kx - y + n$$

transforms $l$ into the $x$-axis, and the point $B$—into the origin. Then without loss of generality it is sufficient to consider a fifth-degree polynomial $f(x)$ with points of inflection $(b, 0)$, $(0, 0)$ and $(a, 0)$, with $b < 0 < a$. Obviously $f''(x) = kx(x-a)(x-b)$, hence

$$f(x) = \frac{k}{20}x^5 - \frac{k(a+b)}{12}x^4 + \frac{kab}{6}x^3 + cx + d\,.$$

By substituting the coordinates of the inflection points, we find $d = 0$, $a + b = 0$ and $c = \frac{7ka^4}{60}$ and therefore

$$f(x) = \frac{k}{20}x^5 - \frac{ka^2}{6}x^3 + \frac{7ka^4}{60}x = \frac{k}{60}\,x(x^2 - a^2)(3x^2 - 7a^2)\,.$$

Since $f(x)$ turned out to be an odd function, the figures bounded by its graph and the $x$-axis are pairwise equiareal. Two of the figures with unequal areas are

$$\Omega_1 :\ 0 \leq x \leq a,\ 0 \leq y \leq f(x); \qquad \Omega_2 :\ a \leq x \leq a\sqrt{\frac{7}{3}},\ f(x) \leq y \leq 0.$$

We find

$$S_1 = S(\Omega_1) = \int_0^a f(x)\,dx = \frac{ka^6}{40}\,,$$

$$S_2 = S(\Omega_2) = -\int_a^{a\sqrt{\frac{7}{3}}} f(x)\,dx = \frac{4ka^6}{405}$$

and conclude that $S_1 : S_2 = 81 : 32$.

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**Problem 3**

Let $\mathbf{SL}_2(\mathbb{Z}) = \{A \mid A \text{ is a } 2\times 2 \text{ matrix with integer entries and } \det A = 1\}$.
a) Find an example of matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^2 + B^2 = C^2$.
b) Show that there do not exist matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^4 + B^4 = C^4$.

**Solution** a) Yes. Example:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

b) No. Let us recall that every $2\times 2$ matrix $A$ satisfies $A^2 - (trA)\,A + (\det A)\,E = 0$ where $trA = a_{11} + a_{22}$.

Suppose that A, B, C $\in \mathbf{SL}_2(\mathbb{Z})$ and $A^4 + B^4 = C^4$. Let $a = trA,\ b = trB,\ c = trC$. Then $A^4 = (aA - E)^2 = a^2A^2 - 2aA + E = (a^3 - 2a)\,A + (1 - a^2)\,E$ and, after same expressions for $B^4$ and $C^4$ have been substituted,

$$\left(a^3 - 2a\right) A + \left(b^3 - 2b\right) B + \left(2 - a^2 - b^2\right) E = \left(c^3 - 2c\right) C + \left(1 - c^2\right) E\,.$$

Calculating traces of both sides we obtain $a^4 + b^4 - 4(a^2 + b^2) = c^4 - 4c^2 - 2$, so $a^4 + b^4 - c^4 \equiv -2 \pmod 4$. Since for every integer $k$: $k^4 \equiv 0 \pmod 4$ or $k^4 \equiv 1 \pmod 4$, then $a$ and $b$ are odd and $c$ is even. But then $a^4 + b^4 - 4(a^2 + b^2) \equiv 2 \pmod 8$ and $c^4 - 4c^2 - 2 \equiv -2 \pmod 8$ which is a contradiction.

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**Problem 4**

Given the real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we define the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by

$$a_{ij} = a_i - b_j \quad \text{and} \quad b_{ij} = \begin{cases} 1, & \text{if } a_{ij} \geq 0, \\ 0, & \text{if } a_{ij} < 0, \end{cases} \qquad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

Consider $C = (c_{ij})$ a matrix of the same order with elements 0 and 1 such that

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} c_{ij}, \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad \sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} c_{ij}, \quad j \in \{1, 2, \ldots, n\}.$$

Show that:

a) $$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = 0 \quad \text{and } B = C.$$

b) $B$ is invertible if and only if there exists two permutations $\sigma$ and $\tau$ of the set $\{1, 2, \ldots, n\}$ such that

$$b_{\tau(1)} \leq a_{\sigma(1)} < b_{\tau(2)} \leq a_{\sigma(2)} < \cdots \leq a_{\sigma(n-1)} < b_{\tau(n)} \leq a_{\sigma(n)}.$$

**Solution**

(a) We have that

$$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n} b_{ij} - \sum_{j=1}^{n} c_{ij} \right) - \sum_{j=1}^{n} b_j \left( \sum_{i=1}^{n} b_{ij} - \sum_{i=1}^{n} c_{ij} \right) = 0. \tag{1}$$

We study the sign of $a_{ij}(b_{ij} - c_{ij})$.

If $a_i \geq b_j$, then $a_{ij} \geq 0$, $b_{ij} = 1$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

If $a_i < b_j$, then $a_{ij} < 0$, $b_{ij} = 0$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

Using (1), the conclusion is that

$$a_{ij}(b_{ij} - c_{ij}) = 0, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}. \tag{2}$$

If $a_{ij} \neq 0$, then $b_{ij} = c_{ij}$. If $a_{ij} = 0$, then $b_{ij} = 1 \geq c_{ij}$. Hence, $b_{ij} \geq c_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$ and since $\sum_{i,j=1}^{n} b_{ij} = \sum_{i,j=1}^{n} c_{ij}$ the final conclusion is that

$$b_{ij} = c_{ij}, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

(b) We may assume that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ since any permutation of $a_1, a_2, \ldots, a_n$ permutes the lines of $B$ and any permutation of $b_1, b_2, \ldots, b_n$ permutes the columns of $B$, which does not change whether $B$ is invertible or not.

- If there exists $i$ such that $a_i = a_{i+1}$, then the lines $i$ and $i+1$ in $B$ are equal, so $B$ is not invertible. In the same way, if there exists $j$ such $b_j = b_{j+1}$, then the columns $j$ and $j+1$ are equal, so $B$ is not invertible.
- If there exists $i$ such that there is no $b_j$ with $a_i < b_j \leq a_{i+1}$, then the lines $i$ and $i+1$ in $B$ are equal, so $B$ is not invertible. In the same way, if there exists $j$ such that there is no $a_i$ with $b_j \leq a_i < b_{j+1}$, then the columns $j$ and $j+1$ are equal, so $B$ is not invertible.
- If $a_1 < b_1$, then $a_1 < b_j$ for any $j \in \{1, 2, \ldots, n\}$, which means that the first line of $B$ has only zero elements, hence $B$ is not invertible.

Therefore, if $B$ is invertible, then $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ separate each other

$$b_1 \leq a_1 < b_2 \leq a_2 < \ldots \leq a_{n-1} < b_n \leq a_n. \tag{3}$$

It is easy to check that if (3), then

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

which is, obviously, invertible.

Concluding, $B$ is invertible if and only if there exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of $a_1, a_2, \ldots, a_n$ and a permutation $b_{j_1}, b_{j_2}, \ldots, b_{j_n}$ of $b_1, b_2, \ldots, b_n$ such that

$$b_{j_1} \leq a_{i_1} < b_{j_2} \leq a_{i_2} < \ldots \leq a_{i_{n-1}} < b_{j_n} \leq a_{i_n}.$$